

STEENROD COALGEBRAS OF SIMPLICIAL COMPLEXES

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ABSTRACT. In this paper, we extend earlier work by showing that if X and Y are ordered simplicial complexes (i.e. simplicial sets whose simplices are determined by their vertices), a morphism $g: N(X) \rightarrow N(Y)$ of Steenrod coalgebras (normalized chain-complexes equipped with extra structure) induces one of topological realizations $\hat{g}: |X| \rightarrow |Y|$. If g is an isomorphism, then it induces an isomorphism between X and Y , implying that $|X|$ and $|Y|$ are homeomorphic.

1. INTRODUCTION

It is well-known that the Alexander-Whitney coproduct is functorial with respect to simplicial maps. If X is a simplicial set, $C(X)$ is the unnormalized chain-complex and RS_2 is the *bar-resolution* of \mathbb{Z}_2 (see [3]), it is also well-known that there is a unique homotopy class of \mathbb{Z}_2 -equivariant maps (where \mathbb{Z}_2 transposes the factors of the target)

$$\xi_X: RS_2 \otimes C(X) \rightarrow C(X) \otimes C(X)$$

cohomology, and that this extends the Alexander-Whitney diagonal. We will call such structures, Steenrod coalgebras and the map ξ_X the Steenrod diagonal. Done carefully (see appendix B in [6]), this Steenrod diagonal is functorial.

In [7], the author defined the functor $\mathcal{C}(\ast)$ on simplicial sets — essentially the chain complex equipped with the structure of a coalgebra over an operad \mathfrak{S} . This coalgebra structure determined all Steenrod and other cohomology operations. Since these coalgebras are not *nilpotent*¹ they have a kind of “transcendental” structure that contains much more information. In [6], the author showed that this transcendental structure even manifests in the sub-operad of \mathfrak{S} generated by $\mathfrak{S}(2) = RS_2$ and proved

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¹In a nilpotent coalgebra, iterated coproducts of elements “peter out” after a finite number of steps. See [4, chapter 3] for the precise definition.

Theorem. *If X and Y are pointed reduced simplicial sets and*

$$f: C(X) \rightarrow C(Y)$$

is a morphism of Steenrod coalgebras — over unnormalized chain-complexes — then f induces a commutative diagram

$$\begin{array}{ccc}
 X & & Y \\
 \uparrow g_X & & \uparrow g_Y \\
 \mathfrak{d} \circ \mathfrak{f}(X) & & \mathfrak{d} \circ \mathfrak{f}(Y) \\
 \downarrow \phi_{(\mathfrak{d} \circ \mathfrak{f}(X))} & & \downarrow \phi_{(\mathfrak{d} \circ \mathfrak{f}(Y))} \\
 \mathbb{Z}_\infty(\mathfrak{d} \circ \mathfrak{f}(X)) & \xrightarrow{f_\infty} & \mathbb{Z}_\infty(\mathfrak{d} \circ \mathfrak{f}(Y)) \\
 \downarrow q_{(\mathfrak{d} \circ \mathfrak{f}(X))} & & \downarrow q_{(\mathfrak{d} \circ \mathfrak{f}(Y))} \\
 \tilde{\mathbb{Z}}(\mathfrak{d} \circ \mathfrak{f}(X)) & \xrightarrow{\tilde{\Gamma}f} & \tilde{\mathbb{Z}}(\mathfrak{d} \circ \mathfrak{f}(Y))
 \end{array}$$

where g_X and g_Y are homotopy equivalences if X and Y are Kan complexes — and homotopy equivalences of their topological realizations otherwise. In particular, if X and Y are nilpotent and f is an integral homology equivalence, then the topological realizations $|X|$ and $|Y|$ are homotopy equivalent.

Here, \mathfrak{f} and \mathfrak{d} are functors defined in definition A.2.

It follows that that the $C(*)$ -functor determines a *nilpotent* space's weak homotopy type. In the present paper, we complement the results of [6] by showing:

Corollary. 4.12. *If X and Y are ordered simplicial complexes, any purely algebraic chain map of normalized chain complexes*

$$f: N(X) \rightarrow N(Y)$$

that makes the diagram

$$\begin{array}{ccc}
 (1.1) & RS_2 \otimes N(X) & \xrightarrow{1 \otimes f} RS_2 \otimes N(Y) \\
 & \downarrow \xi_X & \downarrow \xi_Y \\
 & N(X) \otimes N(X) & \xrightarrow{f \otimes f} N(Y) \otimes N(Y)
 \end{array}$$

commute induces a map of simplicial sets

$$\hat{f}: \mathfrak{d}(X) \rightarrow \mathfrak{d}(Y)$$

which are equipped with canonical inclusions

$$\begin{aligned}\iota_X: X &\rightarrow \mathfrak{d}(X) \\ \iota_Y: Y &\rightarrow \mathfrak{d}(Y)\end{aligned}$$

that induce homeomorphisms of their topological realizations. If f is an isomorphism, then X and Y are isomorphic, hence homeomorphic.

In all cases, the diagram

$$\begin{array}{ccc} H_i(N(X)) & \xrightarrow{g} & H_i(N(Y)) \\ H_i(j_X) \downarrow \cong & & \cong \downarrow H_i(j_Y) \\ H_i(C(\mathfrak{d}(X))) & \xrightarrow{H_*(\hat{g})} & H_i(C(\mathfrak{d}(Y))) \end{array}$$

commutes for all $i \geq 0$, where j_X and j_Y are chain-maps induced by the inclusion of $N(X)$ and $N(Y)$ into the $C(\mathfrak{d}(X))$ and $C(\mathfrak{d}(Y))$, respectively.

Recall that an *ordered simplicial complex* is a simplicial set without degeneracies whose simplices are uniquely determined by their vertices (for instance, a piecewise linear manifold). The proof *requires* X and Y to be ordered simplicial complexes and is likely not true for arbitrary simplicial sets. Also note that we require diagram 1.1 to commute *exactly*, not merely up to a chain-homotopy (as is done when using it to compute Steenrod squares).

This and the main result in [6] imply that old mathematical structures like chain-complexes and Steenrod diagonals encapsulate *vast* amounts of information about a space — and that the traditional ways of studying them (taking cohomology, for example) throw most of this information away.

The author is indebted to Dennis Sullivan for several interesting discussions.

2. DEFINITIONS AND ASSUMPTIONS

Throughout this paper $C(*)$ will denote the unnormalized chain complex and $N(*)$ the normalized one.

If C is a chain-complex

$$(2.1) \quad C^{\otimes n} = \underbrace{C \otimes \cdots \otimes C}_{n \text{ factors}}$$

Definition 2.1. We will denote the category of \mathbb{Z} -free chain complexes by \mathbf{Ch} and ones that are *bounded from below* in dimension 0 by \mathbf{Ch}_0 .

We make extensive use of the Koszul Convention (see [2]) regarding signs in homological calculations:

Definition 2.2. If $f: C_1 \rightarrow D_1$, $g: C_2 \rightarrow D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where a is a homogeneous element), then $(f \otimes g)(a \otimes b)$ is defined to be $(-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)$.

Remark 2.3. If f_i, g_i are maps, it isn't hard to verify that the Koszul convention implies that $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2 \otimes g_1 \circ g_2)$.

The set of morphisms of chain-complexes is itself a chain complex:

Definition 2.4. Given chain-complexes $A, B \in \mathbf{Ch}$ define

$$\mathrm{Hom}_{\mathbb{Z}}(A, B)$$

to be the chain-complex of graded \mathbb{Z} -morphisms where the degree of an element $x \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$ is its degree as a map and with differential

$$\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f$$

As a \mathbb{Z} -module $\mathrm{Hom}_{\mathbb{Z}}(A, B)_k = \prod_j \mathrm{Hom}_{\mathbb{Z}}(A_j, B_{j+k})$.

Remark. Given $A, B \in \mathbf{Ch}^{S_n}$, we can define $\mathrm{Hom}_{\mathbb{Z}S_n}(A, B)$ in a corresponding way.

3. STEENROD COALGEBRAS

We begin with:

Definition 3.1. A *Steenrod coalgebra*, (C, δ) is a chain-complex $C \in \mathbf{Ch}$ equipped with a \mathbb{Z}_2 -equivariant chain-map

$$\delta: RS_2 \otimes C \rightarrow C \otimes C$$

where \mathbb{Z}_2 acts on $C \otimes C$ by swapping factors and RS_2 is the bar-resolution of \mathbb{Z} over $\mathbb{Z}S_2$. A morphism $f: (C, \delta_C) \rightarrow (D, \delta_D)$ is a chain-map $f: C \rightarrow D$ that makes the diagram

$$\begin{array}{ccc} RS_2 \otimes C & \xrightarrow{1 \otimes f} & RS_2 \otimes D \\ \delta_C \downarrow & & \downarrow \delta_D \\ C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \end{array}$$

commute.

Steenrod coalgebras are very general — the underlying coalgebra need not even be coassociative. The category of Steenrod coalgebras is denoted \mathcal{S} .

Remark. It turns out that Steenrod coalgebras are coalgebras over the free operad generated by RS_2 . We will not need this fact in this paper.

Definition 3.2. If

$$\delta: RS_2 \otimes C \rightarrow C \otimes C$$

is a Steenrod coalgebra, the structure map above induces the adjoint structure map

$$(3.1) \quad \alpha: C \rightarrow \text{Hom}_{\mathbb{Z}S_2}(RS_2, C \otimes C) \subset \text{Hom}_{\mathbb{Z}}(RS_2, C \otimes C)$$

Let

$$H_2(C) = \text{Hom}_{\mathbb{Z}}(RS_2, C \otimes C)$$

and inductively define

$$H_n(C) = \text{Hom}_{\mathbb{Z}}(RS_2, H_{n-1}(C) \otimes C)$$

with chain-maps

$$(3.2) \quad \begin{aligned} \alpha_2 &= \alpha: C \rightarrow H_2(C) \\ \alpha_n &= \text{Hom}_{\mathbb{Z}}(1, \alpha_{n-1} \otimes 1) \circ \alpha: C \rightarrow H_n(C) \end{aligned}$$

for all $n \geq 2$.

Proposition 3.3. *Under the hypotheses of definition 3.2, there exist chain-maps*

$$\beta_n: H_n(C) \rightarrow \text{Hom}_{\mathbb{Z}}(RS_2^{\otimes(n-1)}, C^{\otimes n})$$

for all $n \geq 2$. It follows that the adjoint structure map induces a chain-map

$$A: C \rightarrow \prod_{n=2}^{\infty} \text{Hom}_{\mathbb{Z}}(RS_2^{\otimes(n-1)}, C^{\otimes n})$$

Remark. See equation 2.1 for the notation $C^{\otimes n}$.

Proof. The map β_2 is the identity. For larger values of n , the existence of β_n is inductively defined as the composite

$$(3.3) \quad \begin{aligned} \text{Hom}_{\mathbb{Z}}(RS_2, H_{n-1}(C) \otimes C) &\xrightarrow{\text{Hom}_{\mathbb{Z}}(1, \beta_{n-1} \otimes 1)} \\ &\text{Hom}_{\mathbb{Z}}(RS_2, \text{Hom}_{\mathbb{Z}}(RS_2^{\otimes(n-2)}, C^{\otimes(n-1)}) \otimes C) \\ &\xrightarrow{\ell} \text{Hom}_{\mathbb{Z}}(RS_2^{\otimes(n-1)}, C^{\otimes n}) \end{aligned}$$

□

4. SIMPLICES AND COMPLEXES

In this section, we consider properties of Steenrod coalgebras that are topologically derived from simplices and simplicial complexes via the construction in appendix B of [6].

The key result is proposition B.5 of [6], which proves that if $e_n = \underbrace{[(1, 2) | \cdots | (1, 2)]}_{n \text{ terms}} \in \mathbf{RS}_2$ and $x \in N(X)_k$ is the image of a k -simplex, then

$$\xi_X(e_k \otimes x) = \eta_k \cdot x \otimes x$$

where $\eta_k = (-1)^{k(k+1)/2}$ and

$$\xi_X: \mathbf{RS}_2 \otimes N(X) \rightarrow N(X) \otimes N(X)$$

is the Steenrod diagonal (see definition 3.1).

Definition 4.1. If k, m are positive integers, C is a chain-complex, and $E_{2,m} = e_m$ and $E_{k,m} = \underbrace{e_m \otimes \cdots \otimes e_m}_{k-1 \text{ iterations}} \in \mathbf{RS}_2^{\otimes(k-1)}$

$$\rho_m = (\eta_m \cdot E_{2,m}, \eta_m^2 \cdot E_{3,m}, \eta_m^3 \cdot E_{4,m}, \dots) \in \prod_{n=2}^{\infty} \mathbf{RS}_2^{\otimes(n-1)}$$

with $\eta_m = (-1)^{m(m+1)/2}$ (see proposition B.5 of [6]) and define

$$\gamma_m: \prod_{n=2}^{\infty} \mathrm{Hom}_{\mathbb{Z}}(\mathbf{RS}_2^{\otimes(n-1)}, C_m^{\otimes n}) \rightarrow \prod_{n=2}^{\infty} C_m^{\otimes n}$$

via evaluation on ρ_m .

We have

Corollary 4.2. *If X is an ordered simplicial complex and $c \in N(X)_n$ is an element generated by an n -simplex, then the image of c under the composite, Ξ_n ,*

$$N(X)_n \xrightarrow{A} \prod_{k=2}^{\infty} \mathrm{Hom}_{\mathbb{Z}}(\mathbf{RS}_2^{\otimes(k-1)}, C_n^{\otimes k}) \xrightarrow{\gamma_n} \prod_{k=2}^{\infty} N(X)^{\otimes k}$$

is

$$(4.1) \quad \Xi_n(c) = (c, c \otimes c, \dots)$$

Here, $N(X)$ is the (normalized) chain complex of X and the chain-map, A , is defined in proposition 3.3.

Remark. Since Ξ_n is constructed using the Steenrod coalgebra structure of $N(X)$, it is natural with respect to Steenrod coalgebra morphisms. Equation 4.1 is generally only valid for chain-complexes of *simplicial sets* and elements, c , induced by *actual simplices*.

Proof. We claim that

$$\beta_j(c) = (\eta_n^{j-1} \cdot E_{j,n} \mapsto c^{\otimes j}) \in \text{Hom}_{\mathbb{Z}}(\text{RS}_2^{\otimes(j-1)}, N(X)_j^{\otimes j})$$

where we follow the notation of proposition 3.3. When $j = 2$, this follows from proposition B.5 of [6] and the fact that c is the image of an n -simplex. If $j > 2$ it follows from the case $j = 2$ and induction on j :

$$\begin{aligned} \beta_j(c) &= \ell \circ \text{Hom}_{\mathbb{Z}}(1, \beta_{j-1} \otimes 1)(\eta_n \cdot E_{2,n} \mapsto c \otimes c) && \text{by equation 3.3} \\ &= \ell(\eta_n \cdot E_{2,n} \mapsto \beta_{j-1}(c) \otimes c) \\ &= \ell(\eta_n \cdot E_{2,n} \mapsto (\eta_n^{j-2} \cdot E_{j-1,n} \mapsto c^{\otimes(j-1)}) \otimes c) && \text{by induction} \\ &= (\eta_n \cdot E_{2,n} \otimes \eta_n^{j-2} \cdot E_{j-1,n} \mapsto c^{\otimes j}) \\ &= (\eta_n^{j-1} \cdot E_{j,n} \mapsto c^{\otimes j}) && \text{definition 4.1} \end{aligned}$$

If

$$p_j: \prod_{k=2}^{\infty} N(X)^{\otimes k} \rightarrow N(X)^{\otimes j}$$

is the projection, then $p_j(\Xi_n(c)) = c^{\otimes j}$ and the conclusion follows. \square

Lemma C.1 of [6] implies that:

Corollary 4.3. *Let X be a simplicial set and suppose*

$$f: \mathcal{N}^n = N(\Delta^n) \rightarrow N(X)$$

is a Steenrod coalgebra morphism. Then the image of the generator $\Delta^n \in N(\Delta^n)_n$ is zero or a generator of $N(X)_n$ defined by an n -simplex of X .

Remark. As the statement implies, we do not need X to be an ordered simplicial complex in this result.

Proof. Since f is a Steenrod coalgebra morphism, it induces a commutative diagram

$$\begin{array}{ccc} N(\Delta^n) & \xrightarrow{\Xi_{n, N(\Delta^n)}} & \prod_{j=2}^{\infty} N(\Delta^n)^{\otimes j} \\ f \downarrow & & \downarrow \prod_{j=2}^{\infty} f^{\otimes j} \\ N(X) & \xrightarrow{\Xi_{n, N(X)}} & \prod_{j=2}^{\infty} N(X)^{\otimes j} \end{array}$$

Suppose

$$f(\Delta^n) = \sum_{k=1}^t c_k \cdot \sigma_k^n \in N(X)$$

where the σ_k^n are images of n -simplices of X and the $c_k \in \mathbb{Z}$. If $f(\Delta^n)$ is not *equal* to one of the σ_k^n (i.e. if more than one of the c_k is nonzero, or if only one is nonzero but not equal to +1), lemma C.1 of [6] implies that its image under $\Xi_{N(X)}$ in corollary 4.2 is *linearly independent* of the images of the σ_k^n , a contradiction. \square

We also conclude that:

Corollary 4.4. *If $f: N(\Delta^n) \rightarrow N(\Delta^n)$ is*

- (1) *an isomorphism of Steenrod coalgebras in dimension n and*
- (2) *an endomorphism of Steenrod coalgebras in lower dimensions*

then f is the identity map.

Proof. Corollary 4.3 implies that f maps every sub-simplex of Δ^n to one of the same dimension. We may identify a k dimensional sub-simplex of Δ^n with a set of $k+1$ vertices $\{i_0, \dots, i_k\}$ with $i_0 < \dots < i_k$.

We are given that f is an isomorphism in dimension n — i.e. it is bijective. We use downward induction on dimension to show that it is bijective in lower dimensions.

If f is bijective in dimension k , every set of $k+1$ vertices $\{j_0, \dots, j_k\}$ occurs exactly *once* as $f(\Delta_\ell^k)$ for some ℓ . Given any k -simplex, Δ^k , with $f(\Delta^k) = \{j_0, \dots, j_k\}$, the boundary $\partial f(\Delta^k)$ is a linear combination of $k+1$ *distinct* faces — namely all k -element subsets of $\{j_0, \dots, j_k\}$. Since f is a chain-map, $f(\partial \Delta^k)$ must be a linear combination of all k -element subsets of $\{j_0, \dots, j_k\}$. It follows that *every* k -element subset of *every* $k+1$ -element set occurs in $f(\Delta_t^{k-1})$ for some $t = 1, \dots, \binom{n+1}{k}$. The Pigeonhole Principle implies that each such k -element subset occurs exactly *once* in the image of f , so that f is bijective on $k-1$ -simplices.

We conclude that f is an *automorphism* of $N(\Delta^n)$. Now we show that f is the *identity map*:

In dimension 0, let f be a permutation, $\pi: \{0, \dots, n\} \rightarrow \{0, \dots, n\}$ of vertices. If $s = (i_1, i_2)$ with $i_1 < i_2$ is a 1-simplex, $f(s) = (j_1, j_2)$ with $j_1 < j_2$ is a 1-simplex, and

$$f(\partial s) = f(i_1) - f(i_2) = \partial f(s) = (j_1) - (j_2) = (\pi i_1) - (\pi i_2)$$

Given the *signs* of the terms in the boundary, we conclude that $i_1 < i_2 \implies \pi i_1 < \pi i_2$ for all $0 \leq i_1 < i_2 \leq n$ (in other words, π cannot swap the ends of a 1-simplex). This forces π to be the identity permutation. It follows that f is the identity map on 1-simplices.

If $k > 1$, $w = (i_0, \dots, i_k)$ is any k -simplex in Δ^n , and

$$\delta_k = \underbrace{(1 \otimes \dots \otimes \delta) \circ \dots \circ \delta}_{k-1 \text{ factors}}: N(\Delta^n) \rightarrow N(\Delta^n)^{\otimes k}$$

where $\delta: N(\Delta^n) \rightarrow N(\Delta^n) \otimes N(\Delta^n)$ is the Alexander-Whitney diagonal, then the image of $\delta_k(w)$ in

$$N(\Delta^n)^{\otimes k} / (N(\Delta^n)^{\otimes k})_0$$

is

$$Z = (i_0, i_1) \otimes (i_1, i_2) \otimes \dots \otimes (i_{k-1}, i_k) \in N(\Delta^n)_1^{\otimes k}$$

where each edge, (i_t, i_{t+1}) , is the result of a sequence, $F_0 \dots F_{t-1} F_{t+1} \dots F_n$, of face-operations applied to w . Since these edges are mapped via the identity map (by the argument above) $f^{\otimes k}(Z) = Z \in N(\Delta^n)_1^{\otimes k}$, which implies that $f(w)$ has the same vertices as w so $f(w) = w$. It follows that f is the identity map in all dimensions. \square

A similar line of reasoning implies that:

Corollary 4.5. *Let X be an ordered simplicial complex and let*

$$f: N(\Delta^n) \rightarrow N(X)$$

map Δ^n to an n -simplex $\sigma \in N(X)$ defined by the inclusion $\iota: \Delta^n \rightarrow X$. Then

$$f(N(\Delta^n)) \subset N(\iota)(N(\Delta^n))$$

so that $f = N(\iota)$.

Proof. Since X is an ordered simplicial complex, the map ι is an inclusion.

Suppose $\Delta^k \subset \Delta^n$ and $f(N(\Delta^k))_k \subset N(\Delta^k)_k$. Since the boundary of Δ^k is an alternating sum of $k+1$ faces, and since they must map to $k-1$ -dimensional simplices of $N(f(\Delta^k))$ with the same signs (so no cancellations can take place) we must have $f(F_i \Delta^k) \subset N(f(\Delta^k))$ and the conclusion follows by downward induction on dimension. The final statements follow immediately from corollary 4.4. \square

Next, we consider *degeneracies*:

Proposition 4.6. *If $n > m$, then surjective Steenrod-coalgebra morphisms*

$$f: N(\Delta^n) \rightarrow N(\Delta^m)$$

are in a 1-1 correspondence with surjective morphisms

$$\mathbf{f}: \mathbf{n} \twoheadrightarrow \mathbf{m}$$

of ordered sets, where $\mathbf{n} = 0 < \cdots < n$ and $\mathbf{m} = 0 < \cdots < m$. In particular, f is induced by the simplicial map

$$\hat{f}: \Delta^n \rightarrow \Delta^m$$

corresponding to \mathbf{f} .

Proof. Certainly any surjective Steenrod-coalgebra morphism, f , defines a surjective morphism of vertices: $\mathbf{f} = \alpha(f)$. Given \mathbf{f} , corollary 4.3 implies that the m -dimensional sub-simplices of Δ^n can either map to Δ^m (in a *unique* way, by corollary 4.4) or 0. The sets

$$\mathbf{f}^{-1}(0), \dots, \mathbf{f}^{-1}(m)$$

represent sub-simplices of Δ^n , which we can imagine that \mathbf{f} collapses to points — defining a morphism of ordered simplicial complexes and a chain-map. Each possible selection $i_0 \in \mathbf{f}^{-1}(0), \dots, i_m \in \mathbf{f}^{-1}(m)$ defines a unique m -simplex $\Delta_{i_0, \dots, i_m}^m \subset \Delta^n$ for which there is a *unique* Steenrod coalgebra morphism (by corollary 4.4)

$$(4.2) \quad f_{i_0, \dots, i_m}: N(\Delta_{i_0, \dots, i_m}^m) \rightarrow N(\Delta^m)$$

We can define a Steenrod coalgebra morphism

$$f: N(\Delta^n) \rightarrow N(\Delta^m)$$

that sends *each* of these to $N(\Delta^m)$ and all other sub-simplices of Δ^n to 0. We will call this morphism $\beta(\mathbf{f})$.

It is not hard to see that $\mathbf{f} = \alpha \circ \beta(\mathbf{f})$. That $f = \beta \circ \alpha(f)$ follows from the *uniqueness* of the morphisms $\{f_{i_0, \dots, i_m}\}$ in equations 4.2. It follows that α and β define inverse one-to-one correspondences.

The final statement follows from the *uniqueness* of Steenrod-coalgebra morphisms corresponding to a surjective morphism $\mathbf{f}: \mathbf{n} \twoheadrightarrow \mathbf{m}$ and the fact that a simplicial map

$$\hat{f}: \Delta^n \rightarrow \Delta^m$$

induces a Steenrod-coalgebra morphism. □

We finally have:

Proposition 4.7. *Let X be an ordered simplicial complex whose vertices form an ordered set \mathbf{m} (which induces orderings on all of the simplices of X). In addition, let $f: N(\Delta^n) \rightarrow N(X)$ be a Steenrod coalgebra morphism whose restriction to vertices induces a set-map*

$$\mathbf{f}: \mathbf{n} \rightarrow \mathbf{m}$$

with $\mathbf{f}(\mathbf{n}) = \{i_0, \dots, i_k\} \subset \mathbf{m}$. Then

- (1) *the set-map, \mathbf{f} , preserves the order-relation, and*

- (2) if $\{j_0, \dots, j_k\} \subset \mathbf{n}$ is any $k+1$ -element set with $\mathbf{f}(\{j_0, \dots, j_k\}) = \{i_0, \dots, i_k\}$, then $f(N(\Delta_{j_0, \dots, j_k}^k)) = N(\Delta_{i_0, \dots, i_k}^k) \subset N(X)$ — where $\Delta_{j_0, \dots, j_k}^k \subset \Delta^n$ is the k -dimensional sub-simplex defined by $\{j_0, \dots, j_k\}$ and $\Delta_{i_0, \dots, i_k}^k \subset X$ is a k -dimensional simplex with vertices $\{i_0, \dots, i_k\}$.

In particular, the image of f is $N(\Delta_{i_0, \dots, i_k}^k) \subset N(X)$.

Remark. It follows that f is a surjection onto a sub-Steenrod coalgebra of $N(X)$ generated by a single simplex of X .

Proof. We prove this by induction on k . It is clearly true for $k = 0$.

In dimension 1, $\mathbf{f}|_{\{j_0, j_1\}}$ is bijective so that the image of a 1-simplex defined by $(i < j)$ with distinct endpoints must be a 1-simplex ($k < \ell$) in X with distinct endpoints. Since the boundary is $j_1 - j_0$ in $N(\Delta_{j_0, j_1}^1)$ and since f is a *chain-map*, it follows that $i < j \in \mathbf{n}$ implies that $\mathbf{f}(i) \leq \mathbf{f}(j) \in \mathbf{m}$ (compare with the 1-dimensional case in the proof of corollary 4.4).

If $\{j_0, \dots, j_k\} \subset \mathbf{n}$ maps to $\{i_0, \dots, i_k\}$ under \mathbf{f} , then the inductive hypothesis implies that every k -element subset of $\{j_0, \dots, j_k\}$ represents a $k-1$ -dimensional face of Δ^n that maps to a $k-1$ -dimensional simplex of X . In other words the faces of $\Delta_{j_0, \dots, j_k}^k$ map nontrivially to $k-1$ -dimensional simplices of X that are *distinct* because

$$\mathbf{f}|_{\{j_0, \dots, j_k\}}$$

is bijective.

If $\sigma \in N(\Delta_{j_0, \dots, j_k}^k)_k$ represents the sub-simplex $\Delta_{j_0, \dots, j_k}^k \subset \Delta^n$, corollary 4.3 implies that $f(\sigma)$ is either 0 or a generator of $N(X)$ representing a k -simplex, $\Delta_{i_0, \dots, i_k}^k$. The case where $f(\sigma) = 0$ contradicts the fact that f is a chain-map and its *boundary* maps to a nonzero element of $N(X)$ (a linear combination of *distinct* $k-1$ -dimensional simplices of X). \square

We finally arrive at:

Theorem 4.8. *If X is an ordered simplicial complex, any Steenrod-coalgebra morphism*

$$f: N(\Delta^n) \rightarrow N(X)$$

is induced by a simplicial map

$$\hat{f}: \Delta^n \rightarrow X$$

Proof. Proposition 4.7 implies that the image of f is generated by a single simplex of X , and corollary 4.4 and proposition 4.6 imply that this is induced by a simplicial map. \square

We define a complement to the $N(*)$ -functor:

Definition 4.9. Define a functor

$$\mathrm{Shom}_{\mathcal{S}}(\star, *): \mathcal{S} \rightarrow \mathbf{S}$$

to the category of simplicial sets, as follows:

If $C \in \mathcal{S}$, define the n -simplices of $\mathrm{Shom}_{\mathcal{S}}(\star, C)$ to be the Steenrod coalgebra morphisms

$$\mathcal{N}^n \rightarrow C$$

where $\mathcal{N}^n = N(\Delta^n)$ is the normalized chain-complex of the standard n -simplex, equipped with the Steenrod coalgebra structure defined in theorem B.2 of [6].

Face-operations are duals of coface-operations

$$d_i: [0, \dots, i-1, i+1, \dots, n] \rightarrow [0, \dots, n]$$

with $i = 0, \dots, n$ and vertex i in the target is *not* in the image of d_i .

Degeneracies are duals of codegeneracy operators

$$\begin{aligned} s_i: [0, \dots, i, i+1, \dots, n+1] &\rightarrow [0, \dots, n] \\ i &\mapsto i \\ i+1 &\mapsto i \end{aligned}$$

Proposition 4.10. *If X is an ordered simplicial complex there exists a natural inclusion*

$$u_X: X \rightarrow \mathrm{Shom}_{\mathcal{S}}(\star, N(X))$$

Proof. To prove the first statement, note that any simplex Δ^k in X comes equipped with a canonical inclusion

$$\iota: \Delta^k \rightarrow X$$

The corresponding order-preserving map of vertices induces an Steenrod-coalgebra morphism

$$N(\iota): N(\Delta^k) = \mathcal{N}^k \rightarrow N(X)$$

so u_X is defined by

$$\Delta^k \mapsto N(\iota)$$

It is not hard to see that this operation respects face-operations. \square

So, $\mathrm{Shom}_{\mathcal{S}}(\star, N(X))$ naturally contains a copy of X . The interesting question is how much *more* it contains:

Theorem 4.11. *If $X \in \mathbf{SC}$ is an ordered simplicial complex, then*

$$\mathbf{Shom}_{\mathcal{S}}(\star, N(X)) = \mathfrak{d}(X)$$

and the canonical map

$$\iota_X: X \rightarrow \mathbf{Shom}_{\mathcal{S}}(\star, N(X))$$

that sends X to the nondegenerate simplices of $\mathfrak{d}(X)$ induces a homeomorphism

$$H: |X| \rightarrow |\mathfrak{d}(X)|$$

of topological realizations.

Remark. Since $\mathfrak{d}(X)$ is degeneracy-free, its nondegenerate simplices form a sub-complex. The homeomorphism, H , is essentially the identity map.

Proof. Theorem 4.8 implies that

$$\mathbf{Shom}_{\mathcal{S}}(\star, N(X)) = \bigsqcup_{\mathbf{m} \twoheadrightarrow \mathbf{n}} X_n = \mathfrak{d}(X)$$

since Steenrod-coalgebra morphisms between Steenrod coalgebras of simplices are *always* induced by *simplicial maps*. The vertex maps are monomorphisms for the simplices of X *only*, and *proper* surjections (i.e. not 1-1) *only* for the added degenerate simplices. The added degenerate simplices are only subject to the basic identities between face- and degeneracy-operators. The conclusion follows from proposition A.6. \square

This immediately implies

Corollary 4.12. *If X and Y are ordered simplicial complexes, any morphism of Steenrod coalgebras*

$$g: N(X) \rightarrow N(Y)$$

induces a map

$$\hat{g}: \mathbf{Shom}_{\mathcal{S}}(\star, N(X)) = \mathfrak{d}(X) \rightarrow \mathbf{Shom}_{\mathcal{S}}(\star, N(Y)) = \mathfrak{d}(Y)$$

of simplicial sets and a map of topological realizations

$$|X| = |\mathfrak{d}(X)| \xrightarrow{|\hat{g}|} |\mathfrak{d}(Y)| = |Y|$$

where $||$ is topological realization. In addition, the diagram*

$$\begin{array}{ccc} H_i(N(X)) & \xrightarrow{g} & H_i(N(Y)) \\ H_i(j_X) \downarrow \cong & & \cong \downarrow H_i(j_Y) \\ H_i(C(\mathfrak{d}(X))) & \xrightarrow{H_*(\hat{g})} & H_i(C(\mathfrak{d}(Y))) \end{array}$$

commutes for all $i \geq 0$, where j_X and j_Y are chain-maps induced by the inclusion of $N(X)$ and $N(Y)$ into $C(\mathfrak{d}(X))$ and $C(\mathfrak{d}(Y))$, respectively.

If g is an isomorphism, then $|\hat{g}|$ is a homeomorphism.

Proof. A morphism $g: N(X) \rightarrow N(Y)$ induces a morphism of simplicial sets

$$\mathrm{Shom}_{\mathcal{S}}(\star, g): \mathrm{Shom}_{\mathcal{S}}(\star, N(X)) \rightarrow \mathrm{Shom}_{\mathcal{S}}(\star, N(Y))$$

which is an isomorphism (and homeomorphism of topological realizations) if g is an isomorphism. The conclusion follows from theorem 4.11. The chain-maps j_X and j_Y are nothing but the inclusions of the sub-chain-complexes generated by *nondegenerate* simplices — which are well-known to be homology equivalences. \square

APPENDIX A. DELTA COMPLEXES AND SIMPLICIAL SETS

We consider variations on the concept of simplicial set.

Definition A.1. Let Δ_+ be the ordinal number category whose morphisms are order-preserving monomorphisms between them. The objects of Δ_+ are elements $\mathbf{n} = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}$ and a morphism

$$\theta: \mathbf{m} \rightarrow \mathbf{n}$$

is a strict order-preserving map ($i < k \implies \theta(i) < \theta(j)$). Then the category of *delta-complexes*, \mathbf{D} , has objects that are contravariant functors

$$\Delta_+ \rightarrow \mathbf{Set}$$

to the category of sets. The chain complex of a delta-complex, X , will be denoted $N(X)$.

Remark. In other words, delta-complexes are just simplicial sets *without degeneracies*. Note that ordered simplicial complexes are particular types of delta-complexes.

A simplicial set gives rise to a delta-complex by “forgetting” its degeneracy-operators — “promoting” its degenerate simplices to non-degenerate status. Conversely, a delta-complex can be converted into a simplicial set by equipping it with degenerate simplices in a mechanical fashion. These operations define functors:

Definition A.2. The functor

$$\mathfrak{f}: \mathbf{S} \rightarrow \mathbf{D}$$

is defined to simply drop degeneracy operators (degenerate simplices become nondegenerate). The functor

$$\mathfrak{d}: \mathbf{D} \rightarrow \mathbf{S}$$

equips a delta complex, X , with degenerate simplices and operators via

$$(A.1) \quad \mathfrak{d}(X)_m = \bigsqcup_{\mathbf{m} \rightarrow \mathbf{n}} X_n$$

for all $m > n \geq 0$.

Remark. The functors \mathfrak{f} and \mathfrak{d} were denoted F and G , respectively, in [5]. Equation A.1 simply states that we add all possible degeneracies of simplices in X subject *only* to the basic identities that face- and degeneracy-operators must satisfy.

Although \mathfrak{f} promotes degenerate simplices to nondegenerate ones, these new nondegenerate simplices can be collapsed without changing the homotopy type of the complex: although the degeneracy operators are no longer built in to the delta-complex, they still define contracting homotopies.

The definition immediately implies that

Proposition A.3. *If X is a simplicial set and Y is a delta-complex, $C(X) = N(\mathfrak{f}(X))$, $N(\mathfrak{d}(Y)) = N(Y)$, and $C(X) = N(\mathfrak{d} \circ \mathfrak{f}(X))$.*

Definition A.4. A simplicial set, X , is defined to be *degeneracy-free* if

$$X = \mathfrak{d}(Y)$$

for some delta-complex, Y .

Remark. Compare definition 1.10 in chapter VII of [1]²). In a manner of speaking, X is freely generated by the degeneracy operators acting on a basis consisting of the simplices of Y . Lemma 1.2 in chapter VII of [1] describes other properties of degeneracy-free simplicial sets (hence of the functor \mathfrak{d}).

In [5], Rourke and Sanderson also showed that one could give a “somewhat more intrinsic” definition of degeneracy-freeness:

Proposition A.5. *If X is a simplicial set, let $\text{Core}(X)$ consist of the nondegenerate simplices and their faces. This is a delta-complex and there exists a canonical map*

$$c: \mathfrak{d}(\text{Core}(X)) \rightarrow X$$

sending simplices of $\text{Core}(X)$ to themselves in X and degeneracies to suitable degeneracies of them. Then X is degeneracy-free if and only if c is an isomorphism.

² Their definition has a typo, stating that Δ_+ consists of *surjections* rather than *injections*.

Theorem 1.7 of [5] shows that there exists an adjunction:

$$(A.2) \quad \mathfrak{d}: \mathbf{D} \leftrightarrow \mathbf{S}: \mathfrak{f}$$

The composite (the *counit* of the adjunction)

$$\mathfrak{f} \circ \mathfrak{d}: \mathbf{D} \rightarrow \mathbf{D}$$

maps a delta complex into a much larger one — that has an infinite number of (degenerate) simplices added to it. There is a natural inclusion

$$(A.3) \quad \iota_X: X \rightarrow \mathfrak{f} \circ \mathfrak{d}(X)$$

and a natural map (the *unit* of the adjunction)

$$(A.4) \quad g: \mathfrak{d} \circ \mathfrak{f}(X) \rightarrow X$$

The functor g sends degenerate simplices of X that had been “promoted to nondegenerate status” by \mathfrak{f} to their degenerate originals — and the extra degenerates added by \mathfrak{d} to suitable degeneracies of the simplices of X .

Rourke and Sanderson also prove:

Proposition A.6. *If X is a simplicial set and Y is a delta-complex then*

- (1) *$|Y|$ and $|\mathfrak{d}Y|$ are homeomorphic,*
- (2) *the map $|g|: |\mathfrak{d} \circ \mathfrak{f}(X)| \rightarrow |X|$ is a homotopy equivalence, so that $|\iota_Y|: |Y| \rightarrow |\mathfrak{f} \circ \mathfrak{d}(Y)|$ is a homotopy equivalence,*
- (3) *$\mathfrak{f}: HS \rightarrow HD$ defines an equivalence of categories, where HS and HD are the homotopy categories, respectively, of \mathbf{S} and \mathbf{D} . The inverse is $\mathfrak{d}: HD \rightarrow HS$.*

Remark. Here, $|*|$ denotes the topological realization functors for \mathbf{S} and \mathbf{D} .

Proof. The first two statements are proposition 2.1 of [5]. □

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